

A NOTE ON CAKE CUTTING

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The algorithmic aspects of the following problem are investigated: n (≥ 2) persons want to cut a cake into n shares so that every person will get at least $1/n$ of the cake by his own measure and so that the number of cuts made on the cake is minimal. The cutting process is to be governed by a protocol (computer program). It is shown that no deterministic protocol exists which is fair (in a sense defined in the text) and results in at most $n-1$ cuts. An $O(n \log n)$ -cut deterministic protocol and an $O(n)$ -cut randomized protocol are given explicitly and a deterministic fair protocol with 4 cuts for $n=4$ is described in the appendix.

1. Introduction

The following is a well known mathematical puzzle [1]. Two persons own a cake which they want to split into two parts to be allotted between them. The cake may be made of different ingredients with different values to each person, i.e., each person has his own measure for evaluating any given part of the cake. Is there any procedure or ‘protocol’ which will enable the two persons to cut the cake into two pieces such that each person will get at least $\frac{1}{2}$ of the cake by his own measure?

The above problem and several possible generalizations involving n persons, has been studied in the literature by several authors (see references). While the papers of Dubins and Spanier [4] and Stromquist [5] prove the existence of certain solutions to a generalized problem, and are nonconstructive in nature, we would like to consider here the algorithmic aspects of the problem.

A simple solution or ‘protocol’ for the 2-participants problem can be described as follows (see [1]):

One of the persons cuts the cake into two pieces and the other chooses his piece out of the two: the cutter is sure to get at least half the cake if he cuts the cake exactly into 2 equal pieces (by his measure), the other person will also get at least half the cake (by his measure) as he is the one who chooses between the two pieces.

We would like to address the generalized problem where n (≥ 2) persons want to cut a cake so that each participant will get at least $1/n$ of the cake by his own measure. While this problem can be set in a precise and formal way (see [6]) we

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would try to keep the discussion here as informal as possible, without sacrificing mathematical precision.

The solution to this problem would be based on a formal definition of a fair protocol. But the concept of a fair protocol is quite hard to define for the n -person case. We shall avoid the necessity of a full definition of a protocol by basing our proof on some properties such a protocol and the participant's 'winning strategies' (to be defined below) must have. Those properties and partial definitions are summarized below.

By a protocol we understand a computer programmable interactive procedure. It may issue queries to the participants whose answers may affect its future decisions. It may issue instructions to the participants such as: "Cut a piece of the cake according to the following specifications: ..." or "The piece labelled X is allotted to participant A ", etc. The protocol has *no information* on the measures of the various pieces as measured by the different participants. It is assumed also that if the participants obey the protocol, then each participant will end up with his piece of cake after *finitely* many steps.

By a *strategy* of a participant we understand an adaptive sequence of 'moves' compatible with the protocol, that the participant chooses sequentially, when called by the protocol. A *winning* strategy for a given participant is a strategy which will guarantee that he gets at least $1/n$ of the cake (by his own measure) *independently* of the other participants' strategies. A protocol is *fair* if every participant has a winning strategy.

To clarify the above concepts let us describe a fair protocol for the n -participants case when more than $n-1$ cuts are allowed (up to $\frac{1}{2}n(n-1)$ for this protocol). (This solution is attributed in [2] to Banach and Knaster.)

ALLOT(n)

If $n = 1$ **then** allot the (remaining) cake to the
remaining participant and halt.

Let the remaining participants (who have not received
their share yet) be $1, 2, \dots, n$.

Tell participant 1 to cut from the cake a part α
which he is willing to have.
 $i \leftarrow 1$.

For $j = 2$ **to** $n-1$, **begin**

Ask participant j if he agrees that i will get α .
if he disagrees tell j to cut a proper piece of α
which he is willing to get.
Call this new piece α .
 $i \leftarrow j$.

end

Ask n whether he agrees that i will get α .

If he does **then** i gets α , **else** n gets α .

$n \leftarrow n - 1$.

ALLOT($n - 1$)

The reader can easily verify that every participant has a winning strategy, which is to cut ‘honestly’ every time he is required to cut by the protocol and to disagree that another participant be allotted a piece bigger than $1/n$ of the cake (by his own measure). Every participant who adopts this strategy is sure to end up with at least $1/n$ of the cake by his own measure. As for the number of cuts: With K participants still not allotted their share, at most $K - 1$ cuts are needed before one additional participant gets his share. Thus $\frac{1}{2}n(n - 1)$ is the maximal number of cuts needed.

This solution to the n -player game is unattractive since it tends to crumble the cake; in fact it is much more attractive for the allotment of gold dust or wine.

Another interesting algorithmic solution to this problem was proposed by Kuhn [6] who, as mentioned above, also suggested a formalization of the problem in general mathematical terms. The solution of Kuhn has the same ‘complexity’ as the above solution as far as the maximal number of cuts of the cut cake is concerned.

If the protocol is allowed to do the cutting and is trusted by all participants to make ‘straight’ cuts, then the number of cuts may be reduced to $n - 1$ by changing the above procedure so that all the cuts made by the participants are only virtual cuts, made by moving the knife over the cake without cutting, except when a piece of the cake is finally allotted by the protocol to a certain participant, in which case the protocol makes the cut and the resulting piece is taken by the specified participant. This solution is also unattractive since it allows virtual cuts and the protocol must be trusted to make straight cuts (see also [3]).

The authors do not know of any protocol which is deterministic (except for the ordering of the players) and uses less than $O(n \log n)$ cuts, for $n > 4$. A fair protocol for $n = 4$ with 4 cuts only is given in the appendix (it follows from Theorem 9 in the sequel that at least n cuts are necessary, for any n) but we don’t know whether this protocol can be generalized, even for $n = 5$. In Section 2 we present an $O(n \log n)$ -cut protocol. In Section 3 a randomized protocol with an average $O(n)$ cuts is exhibited. In the last section we prove the nonexistence of a deterministic fair protocol for the n -person case ($n \geq 3$) with $n - 1$ cuts only. We conjecture that no fair deterministic protocol exists in which the number of cuts is $O(n)$.

2. An $O(n \log n)$ -cut protocol

The protocol to be presented in this section does not ask any queries: it only issues instructions. The players make ‘statements’ about how they evaluate pieces through the way they cut.

We shall assume here and in Section 3, that the cake is the $[0, 1]$ interval, but clearly, each player may have his own measure defined over the interval. This restriction is needed for the sake of simplicity. The general case will require some additional definitions (concerning the way a cut is made) and some minor changes in the protocol (to prevent cuts from crossing one another). The following protocol P_1 is suggested:

(1) The protocol tells each of the players, $1, 2, \dots, n$, to cut the cake into a left and right parts whose ratio is $\lfloor n/2 \rfloor : \lceil n/2 \rceil$. Each player uses his own measure and ignores the cuts made by the others.

(2) After this is done, the protocol notes the order of the cuts $0 < c_1 \leq c_2 \leq \dots \leq c_n < 1$, where i_1, i_2, \dots, i_n is a permutation of $1, 2, \dots, n$ and cut c_j was made by player i_j .

(3) The cake cutting game is now divided into two separate games. Players $i_1, i_2, \dots, i_{\lfloor n/2 \rfloor}$ play on $[0, c_{\lfloor n/2 \rfloor}]$ while players $i_{\lfloor n/2 \rfloor + 1}, i_{\lfloor n/2 \rfloor + 2}, \dots, i_n$ play on $[c_{\lfloor n/2 \rfloor}, 1]$. The same protocol is used recursively until one player remains in each game, in which case the player gets the whole piece on which the game is played.

A winning strategy is to follow the cutting instructions precisely. The proof of the protocol's fairness is by induction on the number of times each player participates in game divisions:

If a player participates in a game division only once before he gets his share, then either $n = 3$ and he happens to be i_1 , or $n = 2$. In the former case, if he cuts, according to the instructions, in a ratio $1:2$, he will get $\frac{1}{3}$ of the cake, by his measure. In the latter case $n = 2$ and the player is either i_1 or i_2 . If he is i_1 he gets $[0, c_1]$ which is exactly $\frac{1}{2}$ of the cake, by his measure. If he is i_2 he gets $[c_1, 1]$ which is at least $\frac{1}{2}$ of the cake, by his own measure.

If a player participates in more than one game division, consider the first game division. If he is one of $i_1, i_2, \dots, i_{\lfloor n/2 \rfloor}$, then he plays with $\lfloor n/2 \rfloor$ players on a piece which is at least $\lfloor n/2 \rfloor / n$ of the cake, by his measure. By the inductive hypothesis he gets at least $1/\lfloor n/2 \rfloor$ of its value. Thus, his share is at least $1/n$ of the whole cake, by his measure. The proof in the case that he is one of $i_{\lfloor n/2 \rfloor + 1}, i_{\lfloor n/2 \rfloor + 2}, \dots, i_n$ is similar.

Let $f(n)$ denote the number of cuts used by P_1 to divide a cake among n players. Clearly,

$$f(n) = n + f(\lfloor n/2 \rfloor) + f(\lceil n/2 \rceil).$$

It is easy to show that $f(n) \leq 2n \log_2 n$, and thus the total number of cuts is $O(n \log n)$. This is also the time complexity of the protocol.

By issuing queries to the players one cut may be saved each time step (1) of P_1 is called for, to reduce the total number of cuts to $f(n) \leq n \log_2 n$, but the number of cuts remains $O(n \log n)$.

3. A randomized protocol with an average $O(n)$ cuts

The protocol P_2 , to be used in this section is essentially P_1 with the exception that one attempts to save cuts by issuing queries. The first game division is achieved by the following protocol.

$$(1) \quad L \leftarrow \Phi, \quad R \leftarrow \Phi, \quad M \leftarrow \{1, 2, \dots, n\}.$$

$$c_L \leftarrow 0, \quad c_R \leftarrow 1.$$

(2) A member $A \in M$ is chosen randomly and asked to cut the cake in $\lfloor n/2 \rfloor : \lceil n/2 \rceil$ ratio; A must cut at some c , $c_L \leq c \leq c_R$.

(3) The following query is issued to members of $M - \{A\}$: “Do you agree to play a l.h.s. game on $[0, c]$ to be shared by $\lfloor n/2 \rfloor$ players?” If a player answers no, then he must agree to play r.h.s. game on $[c, 1]$ to be shared by $\lceil n/2 \rceil$ players. Let L_c (R_c) be the set of those whose answer is yes (no).

(4) If $|L_c| + |L| \leq \lfloor n/2 \rfloor - 2$, then $L \leftarrow L \cup L_c \cup \{A\}$, $c_L \leftarrow c$, $M \leftarrow R_c$ and go to (2).

(5) If $|R_c| + |R| \leq \lceil n/2 \rceil - 2$, then $R \leftarrow R \cup R_c \cup \{A\}$, $c_R \leftarrow c$, $M \leftarrow L_c$ and go to (2).

(6) [The conditions of (4) and (5) being false implies that $\lceil n/2 \rceil \geq |R \cup R_c| \geq \lceil n/2 \rceil - 1$.] If $|R_c \cup R| = \lceil n/2 \rceil$, then $L \cup L_c \cup \{A\}$ is assigned to the l.h.s. game and $R_c \cup R$ to the r.h.s. game. Else ($|R_c \cup R| = \lceil n/2 \rceil - 1$), $L \cup L_c$ is assigned to the l.h.s. game and $\{A\} \cup R_c \cup R$ to the r.h.s. game.

One easily observes that the subsets of players, L , M , and R , always satisfy the following conditions:

$$(C1) \quad |L| \leq \lfloor n/2 \rfloor - 1,$$

$$(C2) \quad |R| \leq \lceil n/2 \rceil - 1,$$

$$(C3) \quad L \cap R = \emptyset,$$

$$(C4) \quad M = \{1, 2, \dots, n\} - (L \cup R), \text{ and thus } |M| \geq 2.$$

(C5) Every member of L has indicated, through the cut he performed or through a query answered, that he will be happy to play in a l.h.s. game on $[0, c_L]$.

(C6) Every member of R has indicated that he will be happy to play in a r.h.s. game on $[c_R, 1]$.

(C7) Every member of M has indicated that he will be happy to play in a l.h.s. game on $[0, c_R]$ or a r.h.s. game on $[c_L, 1]$.

The comment made in the beginning of step (6) is also easily proved as follows: The condition of step (4) being false implies that $|L_c| + |L| \geq \lfloor n/2 \rfloor - 1$. Thus,

$$|R_c| + |R| = n - [|L_c| + |L| + 1] \leq n - \lfloor n/2 \rfloor = \lceil n/2 \rceil.$$

The condition of step (5) being false implies that $|R_c| + |R| \geq \lceil n/2 \rceil - 1$, and the comment follows.

One observes that the first game division is achieved in at most $n - 1$ steps, and thus, as in the previous section, the protocol uses $O(n \log n)$ cuts.

However, we shall show that due to the randomization, the expected number of

cuts until a game division is achieved is in fact $O(\log n)$. If we denote by $f(n)$ the expected number of cuts to achieve a complete allotment, then for some constant k ,

$$f(n) \leq k \log n + f(\lfloor n/2 \rfloor) + f(\lceil n/2 \rceil). \quad (1)$$

It is easy to verify that $f(n) \leq an + k \log n - 2k$ satisfies this recursive inequality for every a which satisfies the inductive basis. Thus, the expected number of cuts of the randomized protocol P_2 is $O(n)$.

Let us now analyze the question of the expected number of cuts, of the randomized protocol, until a game division is reached.

We assume that player i , if asked to cut the cake in $\lfloor n/2 \rfloor : \lceil n/2 \rceil$ ratio, will cut at c_i , and that all his answers to queries are determined by c_i as follows: Let c be some cut. If player i is asked whether he agrees to join the l.h.s. game on $[0, c]$ or not, he will answer yes if $c_i \leq c$ and will answer no if $c < c_i$.

Assume, the dividing (virtual) cuts, as above, of the various players are $0 < c_{j_1} \leq c_{j_2} \leq \dots \leq c_{j_n} < 1$. Once cut c_{j_k} , for $k = \lfloor n/2 \rfloor$ or $\lfloor n/2 \rfloor + 1$ is made, a game division is reached. Thus, the situation is very much like that of a median search [7], under the following condition: When an element c of $\{c_1, c_2, \dots, c_n\}$ is picked, we are told which c_i 's are below it and which are above it. Our purpose is to find c_{j_k} .

The randomized protocol for game division is similar to the following procedure for finding the r -ranked element in a set D of m numbers:

RANK(r, m, D) : [$1 \leq r \leq m, |D| = m$].

(0) **If** $m = 1$ **return** the unique element of D .

(1) Choose $d \in D$, randomly. Let $L, R \subset D$, where $L \cap R = \emptyset$.
 $D - \{d\} = L \cup R$, $d_i \in L$ implies $d_i \leq d$ and $d_j \in R$ implies $d_j \geq d$.

(2) **If** $|L| \leq r - 2$ **then begin** $r \leftarrow r - |L| - 1$
 $m \leftarrow m - |L| - 1$
 $D \leftarrow D - L - \{d\}$

end

(3) **If** $|R| \leq m - r - 1$ **then begin** $m \leftarrow m - |R| - 1$
 $D \leftarrow D - R - \{d\}$
RANK(r, m, D)

end

(4) [$|L| = r - 1$ and $|R| = m - r$] **Return** d .

Actually, our game division protocol is slightly more efficient than the above procedure, since it finds an element whose rank is either $\lfloor n/2 \rfloor$ or $\lfloor n/2 \rfloor + 1$, but for our purposes, this extra efficiency is insignificant.

Let us denote by $g(k, n)$ the expected number of applications of step (1) of **RANK**($k, n, \{c_1, c_2, \dots, c_n\}$), where $1 \leq k \leq n$ (and is not necessarily $\lfloor n/2 \rfloor$); i.e.,

$g(k, n)$ is the expected number of times elements are chosen randomly before the k -ranked element is found.

It is easy to see that for $n > 1$

$$g(k, n) = 1 + \frac{1}{n} \left[\sum_{\lambda=1}^{k-1} g(k-\lambda, n-\lambda) + \sum_{p=1}^{n-k} g(k, n-p) \right]. \quad (2)$$

while $g(1, 1) = 0$.

Lemma 1. For $n > 1$,

$$g(1, n) = 1 + \sum_{i=3}^n \frac{1}{i}.$$

Proof. The lemma is obviously true for $n = 2$.

By (2), for $n \geq 2$

$$g(1, n) = 1 + \frac{1}{n} \sum_{p=1}^{n-1} g(1, n-p),$$

$$g(1, n+1) = 1 + \frac{1}{n+1} \sum_{p=1}^n g(1, n+1-p).$$

Thus,

$$(n+1)g(1, n+1) - ng(1, n) = 1 + g(1, n),$$

or

$$g(1, n+1) = g(1, n) = \frac{1}{n+1}.$$

and the lemma follows by induction. \square

Lemma 2. For $n \geq k+1$,

$$g(k, n+1) - g(k, n) = \frac{1}{n-k+2}.$$

Proof. By induction on k . The basis, $k = 1$, has been shown in the proof of Lemma 1. By (2),

$$ng(k, n) = n + \sum_{\lambda=1}^{k-1} g(k-\lambda, n-\lambda) + \sum_{p=1}^{n-k} g(k, n-p),$$

and

$$(n+1)g(k, n+1) = (n+1) + \sum_{\lambda=1}^{k-1} g(k-\lambda, n+1-\lambda) + \sum_{p=1}^{n+1-k} g(k, n+1-p).$$

Thus,

$$\begin{aligned} (n+1)g(k, n+1) - ng(k, n) &= 1 + \sum_{\lambda=1}^{k-1} [g(k-\lambda, n+1-\lambda) - g(k-\lambda, n-\lambda)] \\ &\quad + \sum_{i=k}^{n-1} [g(k, i) - g(k, i)] + g(k, n). \end{aligned}$$

For $k > 1$, by the inductive hypothesis,

$$(n+1)[g(k, n+1) - g(k, n)] = 1 + \sum_{\lambda=1}^{k-1} \frac{1}{n-k+2}.$$

or

$$(n+1)[g(k, n+1) - g(k, n)] = \frac{n-k+2+k-1}{n-k+2} = \frac{n+1}{n-k+2},$$

and the lemma follows. \square

Lemma 3. For $n > 2$,

$$g(2, n) = \frac{5}{3} + \sum_{i=3}^{n-1} \frac{1}{i}.$$

Proof. It is easy to see that $g(2, 2) = 1$ (deterministically).

Let us compute $g(2, 3)$ by (2).

$$g(2, 3) = 1 + \frac{1}{3}[g(1, 2) + g(2, 2)] = \frac{5}{3}.$$

For $n \geq 4$ we can apply Lemma 2:

$$\begin{aligned} g(2, n) &= g(2, n-1) + \frac{1}{n-1} \\ &= g(2, n-2) + \frac{1}{n-2} + \frac{1}{n-1} \\ &\dots \\ &= g(2, 3) + \sum_{i=3}^{n-1} \frac{1}{i}. \quad \square \end{aligned}$$

Lemma 4. $g(k, n) = g(n-k+1, n)$.

The lemma follows from the inherent symmetry of RANK.

Corollary.

$$g(k, k+1) = g(2, k+1) = \frac{5}{3} + \sum_{i=3}^k \frac{1}{i}.$$

Lemma 5. For $n > k \geq 2$,

$$g(k, n) = \frac{5}{3} + \sum_{i=3}^k \frac{1}{i} + \sum_{i=3}^{n-k+1} \frac{1}{i}.$$

Proof. By Lemma 2, for $n > k+1$

$$g(k, n) = g(k, n-1) + \frac{1}{n-k+1}$$

$$\begin{aligned}
&= g(k, n-2) + \frac{1}{n-k} + \frac{1}{n-k+1} \\
&\dots \\
&= g(k, k+1) + \frac{1}{3} + \frac{1}{4} + \dots + \frac{1}{n-k+1} \\
&= \frac{5}{3} + \sum_{i=3}^k \frac{1}{i} + \sum_{i=3}^{n-k+1} \frac{1}{i}. \quad \square
\end{aligned}$$

Theorem 6. *The expected number of cuts made for cutting a cake, by the randomized protocol P_2 is $O(n)$.*

Proof. Since $\sum_{i=2}^n 1/i < \ln n$, by Lemma 5, the expected number of cuts made before a game division is reached, for n players, is $O(\log n)$. By (1) and the discussion which follows it, the theorem holds (with multiplying constant < 2). \square

4. The nonexistence of a deterministic protocol for $n \geq 3$

Our purpose is to show that no fair deterministic protocol exists for the n -persons case, $n \geq 3$, in which exactly $n-1$ cuts are made, even if all participants have *identical measures* for evaluating pieces of the cake, an assumption to be understood in the sequel.

First we prove two lemmas:

Lemma 7. *If there is a protocol as above and if all participants follow their winning strategies, then all cuts made divide the cake into multiples of $1/n$ of the cake so that when the cutting process terminates the cake is cut into n equal pieces.*

Proof. All participants, following their winning strategies, will get at least $1/n$, and therefore exactly $1/n$, of the cake when the process terminates, and all the measures are assumed to be identical. \square

Lemma 8. *Assume there is a fair protocol as described above. If in the course of the cutting, some piece is cut by participant $B \neq A$, into two pieces, then A 's winning strategy allows him to designate one of the two pieces and make sure that he will not end up with it or with any proper part of it.*

Proof. The lemma must hold true for otherwise A could be forced to take some piece which is less than $1/n$ of the cake, cut by some participant B , even if he acts 'honestly' and according to his winning strategy. \square

Theorem 9. *A fair protocol, as described above, does not exist.*

Proof. Assume such a fair protocol P exists. Assume that n persons act according to it, each following his own winning strategy. By Lemma 7, each of the pieces during the cutting process is an integral multiple of $1/n$.

Consider the first cut after which each piece is at most $2/n$ in size. Assume that A is the participant to make this cut (in accordance with the protocol's request). Clearly, he either cuts a $4/n$ piece in the middle, or he cuts a $3/n$ piece into $1/n$ and $2/n$. Assume that while he makes this cut he goes berserk. We will show that the other participants can no longer use their winning strategies so as to assure themselves a fair share.

Case 1. A 's cut is of a $3/n$ piece, and he cuts $(1+\varepsilon)/n$ and $(2-\varepsilon)/n$ where $0 < \varepsilon < 1$. The other participants must be able, by using their winning strategies, to force A into sharing the $3/n$ piece cut by him: The $3/n$ piece must be cut into at least 3 pieces. (Otherwise at least 2 pieces of size less than $1/n$ will be created depriving some participant other than A of his fair share.) But, no matter how the $3/n$ piece is cut again one of its parts will be smaller than $1/n$ of the cake and they must be able to avoid taking that part. Now if A is allowed to cut again he may cut either the $(1+\varepsilon)/n$ piece or the $(2-\varepsilon)/n$ piece (as the case may be) into two equal pieces, both smaller than $1/n$ of the cake, thus depriving some other participant of his fair share. On the other hand if $B \neq A$ makes the additional cut then, by Lemma 8, A can make sure that he does not end up with the smaller of the 2 pieces created by B (the smaller piece will necessarily be smaller than $1/n$ of the cake) and again one other participant is deprived of his fair share.

Case 1. A 's cut is of a $4/n$ piece, and he cuts $(2-\varepsilon)/n$ and $(2+\varepsilon)/n$ where $0 < \varepsilon < 1$. Using an argument similar to the one used in Case 1, it is clear that here too, the other participants must be able, by using their winning strategies, to force A into sharing the $4/n$ piece cut by him, together with some other, 3 participants. Now if A is allowed to cut again and he cuts the $(2-\varepsilon)/n$ piece he may cut it into 2 equal parts thus depriving some other participant of his fair share. If A cuts the $(2+\varepsilon)/n$ piece he may cut an $1/n$ piece out of it thus creating a situation identical to Case 1 (one $(2-\varepsilon)/n$ piece and one $(1+\varepsilon)/n$ piece to be shared by 3 participants). On the other hand if $B \neq A$ makes the cut then by Lemma 8 A can make sure that he does not end up with a piece smaller than $1/n$ of the cake, cut by B . Thus, if B cuts the $(2-\varepsilon)/n$ piece into two then at least one participant, other than A , will be deprived of his fair share. On the other hand, if B cuts the $(2+\varepsilon)/n$ piece into 2 parts of size a and b , with $a \leq b$, then A can reject the smaller piece. If $a < 1/n$, then somebody else will be deprived of his fair share while if $(2+\varepsilon)/2n \leq b \leq (1+\varepsilon)/n$, then again, A cannot be forced to accept a part of b smaller than $1/n$ without depriving somebody else of his fair share.

The proof is now complete.

Remark. An even simpler proof of our theorem can be given if the assumptions are weakened to allow different measures for different participants.

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Appendix

A fair protocol for 4 participants with up to 4 cuts allowed. Label the participants: a, b, c, d ; assume that the cake has size 1.

1. Let a cut the cake into two equal halves (by a 's measure) to be labelled X and Y .
2. Let b, c, d evaluate X and Y by their measures.
 - 2.1. $w_b(X) \geq \frac{1}{2}$; $w_c(Y)$, $w_d(Y) \geq \frac{1}{2}$ ($w_b(X)$ = the size of X as measured by b , etc.) then b and a share X , c and d share Y , done! All similar cases are dealt with in a similar way. Or
 - 2.2. $w_b(Y)$, $w_c(Y)$, $w_d(Y) > \frac{1}{2}$ ($w_b(X)$, $w_c(X)$, $w_d(X) < \frac{1}{2}$). (The other symmetric case is dealt with in a similar way.) Then
3. Let a cut X into 2 equal parts, by his own measure, to be labelled I and II.
4. Let b, c, d evaluate I and II by their measures.
 - 4.1. If one of b, c, d evaluates either I or II as bigger or equal to $\frac{1}{4}$, then that participant gets that piece; a gets the other piece, the remaining participants share Y , done! Or
 - 4.2. Both I and II are evaluated by all three participants as less than $\frac{1}{4}$. At least two participants, say b and c , evaluate one of the 2 pieces I and II, say II, as bigger than the other: $\frac{1}{4} > w_b(II) \geq w_b(I)$; $\frac{1}{4} > w_c(II) \geq w_c(I)$; $\frac{1}{4} > w_d(II)$, $w_d(I)$; ($\frac{1}{4} = w_d(I) = w_d(II)$). Let a get I and
5. Let b cut Y into 2 equal pieces, by b 's measure to be labelled III and IV.

6. Let c and d evaluate III and IV. *Comment:* The following information is available to the protocol after step 5 has been executed.

(i) Due to $\frac{1}{4} > w_b(\text{II}) \geq w_b(\text{I})$ and $w_b(\text{II}) = w_b(\text{IV}) > \frac{1}{2}$ and $w_b(\text{I} + \text{II} + \text{III} + \text{IV}) = 1$, b must agree to share any of the three combinations II + III, II + IV, III + IV with some other participant.

(ii) As $w_c(\text{I}) < \frac{1}{4}$, if $w_c(\text{III}) < \frac{1}{4}$ (or $w_c(\text{IV}) < \frac{1}{4}$), then $w_c(\text{II} + \text{IV}) > \frac{1}{2}$ (or $w_c(\text{II} + \text{III}) > \frac{1}{2}$).

(iii) Same as (ii) for d .

The continuation of the protocol depends on the result of step 6. Let $-$ denote $< \frac{1}{4}$ and let $+$ denote $\geq \frac{1}{4}$. There are 3 possible combinations for the evaluations of c ($w_c(\text{III}) = -$ together with $w_c(\text{IV}) = -$ is impossible as $w_c(\text{III} + \text{IV}) > \frac{1}{2}$) and 3 possible different evaluations for d but only 5 of the possible combined evaluations are distinct and they are listed in the table below, together with the decision of the protocol.

6.1. Table

	III	IV	Decision
w_c	$-$	$+$	d gets III, b and c will share
w_d	$+$	$-$	II + IV (see (i) and (ii) above)
w_c	$-$	$+$	b gets III, c and d will share II + IV
w_d	$-$	$+$	(see (i) and (ii) and (iii) above)
w_c	$-$	$+$	d gets III, b and c will share II + IV
w_d	$+$	$+$	(see (i) and (ii))
w_c	$+$	$+$	c gets III, b and d will share II + IV
w_d	$-$	$+$	
w_c	$+$	$+$	Assume also that $w_c(\text{IV}) > w_c(\text{III})$. Then
w_d	$+$	$+$	d gets III, b and c will share II + IV. <i>Comment:</i> $w_c(\text{IV}) \geq w_c(\text{II})$ and $w_c(\text{II}) \geq w_c(\text{I})$ imply that $w_c(\text{II} + \text{IV}) \geq \frac{1}{2}$. As for b see (i) above.

Done!